Dynamics of a charged particle in a linearly polarized traveling electromagnetic wave

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The relativistic motion of a charged particle in a linearly polarized homogeneous electromagnetic wave is studied using the Hamiltonian formalism. First, a single particle in a linearly polarized traveling wave propagating in a nonmagnetized space is studied. It is shown that the charged particle can have a high average velocity along the propagation direction of the wave. The same result is derived considering an electromagnetic wave in a cold electron plasma. The case of a traveling wave propagating along a constant homogeneous magnetic field is then considered and shown to be nonintegrable. Using canonical transformations, it is shown that the equations of motion can be derived from an autonomous Hamiltonian and a formal solution is found for all the variables of the system. Considering that the wave propagates in vacuum and the particle is initially resonant and at rest, a set of equations is found coupling the energy of the particle and the phase of the wave. Then, the expression for the energy and the differential equation for the phase allow a solution in terms of quadratures. Finally, asymptotic solutions for the phase, the energy and consequently all of the variables are found.

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I. INTRODUCTION

The study of the relativistic dynamics of charged particles in electromagnetic fields is of prime importance in plasma and accelerator physics [1,2]. We hope to find new ways to accelerate charged particles with microwaves and explore new processes in the field of laser-matter interaction in order to generate strong magnetic fields in laser targets. In particular, it is shown in a simple case how Hamiltonian dynamics can help to predict phenomena which take place in plasmas interacting with a relativistically strong wave or in low density plasmas interacting with a moderate intensity wave.

First, the dynamics of one particle in a linearly polarized traveling wave propagating in a nonmagnetized vacuum is considered. Integrability is demonstrated by showing that this time-dependent Hamiltonian system possesses three independent invariants in involution. An extension of the definition of integrability given for autonomous systems is applied. In this case no chaos can take place. In this sense, we say that Liouville's theorem on integrability still holds in the case of time-dependent Hamiltonian systems [3]. Complete integrability is also proven demonstrating that trajectories are on a torus and that the state of the system can be expressed in terms of canonical action-angle variables [4–8]. The question concerning the integrability of the motion of charged particles is not only an academic problem. For instance, in the case of a practical device like the free-electron laser, an experiment performed some time ago at the Massachussetts Institute of Technology [9] was partly explained by showing that trajectories of electrons become chaotic under certain conditions [10,11]. It is shown that the charged particle can have a high average velocity along the direction of propagation of the wave. Thus an electromagnetic wave could create a constant electron current in a plasma. Indeed, considering a cold electron plasma, it is shown that a strong linearly polarized electromagnetic wave can generate a constant electron current along its direction of propagation when the wave has a phase velocity very close to the speed of light in vacuum [12]. Because of relativistic effects, this can be true when the intensity of the wave is very high and/or when the plasma has a very low density compared to the nonrelativistic critical density.

The dynamics of a charged particle in a linearly polarized electromagnetic wave propagating along a constant homogeneous magnetic field is studied next. This part is rather connected to the physics of cyclotron accelerators. This problem has already been explored by Roberts and Buchsbaum in the case of a circularly polarized wave. They found a ''synchronous'' solution in which the particle gains energy indefinitely. This solution occurs because the particle gains energy parallel to, as well as perpendicular to, the propagation direction of the circularly polarized plane wave. The increase in perpendicular energy lowers the cyclotron frequency of the charged particle, while the increase in parallel energy changes the velocity of the particle, resulting in a Doppler shift to a lower frequency as seen by the particle. In this case, the Doppler shift to the lower frequency equals the reduction in the cyclotron frequency and the particle remains ''synchronously'' in cyclotron-resonance condition [13]. It is shown numerically that the synchronous solution still exists when the wave is linearly polarized [14]. A canonical transformation [4–8,15] can change this system into an autonomous one with three degrees of freedom. Only two constants, independent and in involution, were found. In vacuum, one of those constants appears in the resonance condition [16]. This means that when a particle is initially resonant it remains resonant forever. Chaotic trajectories were evidenced by performing Poincaré maps. Then by using Hamilton's equations, it is found that the different variables describing

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the system can be formally given in terms of quadratures involving the energy of the charged particle and the phase of the wave. When the wave propagates in vacuum and the resonance condition is satisfied, an expression defining the energy as a function of the phase of the wave and a differential equation for the phase are derived. These two equations and their formal solution allow us to derive a solution for the system in terms of quadratures. Finally, an asymptotic solution for the energy, and consequently all of the variables of the system, is found.

II. DYNAMICS OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC LINEARLY POLARIZED TRAVELING WAVE

A. Dynamics of one particle only

The components of the electromagnetic field are

$$E_x = E_0 \sin(\omega_0 t - k_0 z), \quad E_y = 0, \quad E_z = 0,$$
$$B_x = 0, \quad B_y = \frac{k_0 E_0}{\omega_0} \sin(\omega_0 t - k_0 z), \quad B_z = 0, \tag{1}$$

where E_0, B_0, ω_0, and k_0 are constants. The following vector potential is chosen

$$\mathbf{A} = \frac{E_0}{\omega_0} \cos(\omega_0 t - k_0 z) \hat{\mathbf{e}}_x . \tag{2}$$

The scalar potential is assumed to vanish.

The Hamiltonian formulation of this problem is derived by using the Lagrangian procedure where time t is treated as a parameter entirely distinct from the spatial coordinates. This Lagrangian formulation is not manifestly Lorentz covariant and is relativistic in the sense that it gives the correct equations [15]. Using this simple procedure, the following relativistic Hamiltonian for a charged particle in the wave is derived in mks units

$$H = \left[\left(P_x + \frac{eE_0}{\omega_0} \cos(\omega_0 t - k_0 z) \right)^2 c^2 + P_y^2 c^2 + P_z^2 c^2 + m^2 c^4 \right]^{1/2}, \tag{3}$$

where $-e$, m, and the $P_i (i = x,y,z)$ are respectively the particle's charge, its rest mass, and its canonical momentum components. This system has three degrees of freedom. P_x, P_y, are constants. One can notice that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\omega_0}{k_0} \frac{\partial H}{\partial z} = \frac{\omega_0}{k_0} \dot{P}_z . \tag{4}$$

As a consequence, a third constant of motion is obtained by integrating this equation

$$C = H - \frac{\omega_0}{k_0} P_z . \tag{5}$$

When considering an n-dimensional autonomous system the existence of n constants of motion means that the phase space trajectories are confined to some n-dimensional manifold in the $2n$-dimensional phase space. The fact that the constants are in involution is used to show that the n-dimensional manifold is a torus. *The existence of these tori in phase space provides the means of defining action variables in an invariant way and allows to integrate the system* [8]. It can be readily checked that P_x, P_y, and C are three independent constants in involution, i.e., their Poisson brackets [4–8] with each other vanish. As a consequence our system is integrable [3].

Let us now introduce the following dimensionless variables and parameters:

$$\hat{z} = k_0 z, \quad \hat{P}_{x,y,z} = \frac{P_{x,y,z}}{mc}, \quad \hat{t} = \omega_0 t,$$
$$\hat{H} = \gamma = \frac{H}{mc^2}, \quad a = \frac{eE_0}{mc\omega_0}. \tag{6}$$

The normalized Hamiltonian of the charged particle is

$$\hat{H} = \{[\hat{P}_x + a \cos(\hat{t} - \hat{z})]^2 + \hat{P}_y^2 + \hat{P}_z^2 + 1\}^{1/2}. \tag{7}$$

A canonical transformation is introduced $(\hat{z}, \hat{P}_z) \rightarrow (\phi, \hat{P}_z)$, given the type-2 generating function [4–8]

$$F_2(\hat{z}, \hat{P}_z) = \hat{P}_z (\hat{z} - \hat{t}). \tag{8}$$

It remains $\hat{P}_z$ unchanged and yields $\phi = \hat{z} - \hat{t}$. The Hamiltonian expressed in terms of the new variables is

$$\hat{H} = \hat{C} = [(\hat{P}_x + a \cos \phi)^2 + \hat{P}_y^2 + \hat{P}_z^2 + 1]^{1/2} - \hat{P}_z , \tag{9}$$

where $\hat{C} = C/mc^2$.

This Hamiltonian, $\hat{P}_x$ and $\hat{P}_y$ are three constants of motion, which are independent and in involution. As a consequence the system is completely integrable.

First, we assume that $\hat{P}_x = \hat{P}_y = 0$ and $\hat{P}_z = 0$ when $\phi = 0$. In this case, the initial Lorentz factor γ_0 equals $\sqrt{1 + a^2}$, the particle is not initially at rest and has the following initial velocity along the x axis

$$\hat{v}_{x0} = \pm \frac{a}{\sqrt{1 + a^2}}. \tag{10}$$

Still, the particle is at rest on average in the x-y plane.

The constant $\hat{C}$ allows us to calculate $\hat{P}_z$

$$\hat{P}_z = \frac{-a^2 \sin^2 \phi}{2\sqrt{1 + a^2}}. \tag{11}$$

Then the velocity of the particle along the z axis normalized to the speed of light is

$$\hat{v}_z = \frac{v_z}{c} = -\frac{a^2 \sin^2 \phi}{2 + a^2 (1 + \cos^2 \phi)}. \tag{12}$$

Then, for any value of a, averaging over ϕ leads to the following average value of $\hat{v}_z$

$$\langle \hat{v}_z \rangle \approx -\frac{a^2}{\pi(2+a^2)} B\left(\frac{3}{2},\frac{1}{2}\right) F\left[\frac{1}{2},1;2;-\frac{a^2}{(2+a^2)}\right], \tag{13}$$

where B is the beta function and F the hypergeometric function. This expression is not very helpful from a practical point of view and one has to deduce features of the motion by deriving approximate expressions in interesting cases. First, considering that a is small compared to unity, one finds $\langle \hat{v}_z \rangle \approx -a^2/4$. For large values of a, one obtains $\langle \hat{v}_z \rangle \approx -(\sqrt{2}-1) \approx -0.41$. The good agreement between the numerical solution for $\hat{v}_z$ derived through the exact equations of motion and these approximate solutions has been verified.

Let us now assume that $\hat{P}_x = -a$, $\hat{P}_y = 0$, and $\hat{P}_z = 0$ when $\phi = 0$. In this case, the particle is supposed to be initially at rest, but it has a drift velocity along the x axis. One obtains

$$\hat{P}_z = \frac{a^2}{2} (\cos\phi - 1)^2. \tag{14}$$

The velocity of the particle along the z axis normalized to the speed of light is

$$\hat{v}_z = \frac{a^2(\cos\phi - 1)^2}{2 + a^2(\cos\phi - 1)^2}. \tag{15}$$

For small values of a, the average value of this velocity is $\langle \hat{v}_z \rangle \approx 3a^2/4$. For large a (large values of the electric field), one has $\langle \hat{v}_z \rangle \approx 1$. The numerical solution of the exact equations of motion shows that the approximate solutions for these average particle velocity along the z axis are appropriate for small and large values of a, respectively. Moreover, when the electric field has a very high magnitude, as the velocity remains close to the speed of light on an average, it can only vary very slowly. This is a consequence of the fact that the particle has a very high energy compared to mc^2.

One can also consider that the traveling wave is obliquely incident with respect to the z axis. The wave vector $\mathbf{k}_0$ is assumed to be in the x-z plane while the electric field is perpendicular to this plane. In the laboratory frame (L), the charged particle is at rest before interacting with the wave. One can then perform a simplifying Lorentz transformation. A new frame (L') is introduced which moves uniformly in the x direction with the velocity U relative to (L). If α and α' are respectively the angles between the wave vector and the z axis in the two frames, the relativistic transformation formulas for the velocity [17] yield a set of two equations between the two angles which show that there is a particular frame (L') for which $\alpha' = 0$. The velocity of this frame along the x axis is defined by $\sin\alpha = U/c$. If we choose to stand in such a frame (L'), then electromagnetic field can be assumed to have the following components

$$E'_x = E'_0 \sin(\omega'_0 t' - k'_0 z'), \quad E'_y = 0, \quad E'_z = 0,$$

$$B'_x = 0, \quad B'_y = \frac{k'_0 E'_0}{\omega'_0} \sin(\omega'_0 t' - k'_0 z'), \quad B'_z = 0. \tag{16}$$

The angle α can be chosen such that

$$\frac{U}{c} = \frac{a'}{\sqrt{1+a'^2}}. \tag{17}$$

where $a' = eE'_0/mc\omega'_0$. In the considered situation, the Lorentz transformation formulas for the field yield $E_0 = \Gamma E'_0$ where $\Gamma = 1/\sqrt{1 - U^2/c^2}$. The law of transformation of the incident wave four-vector gives $\omega_0 = \Gamma \omega'_0$. Letting

$$a = e\sqrt{(E_{0x}^2 + E_{0z}^2)}/mc\omega_0, \tag{18}$$

it can be very easily shown that $a' = a$. Consequently, the velocity of the frame (L') along the x axis is given, in function of quantities defined in the laboratory frame, by

$$\frac{U}{c} = \frac{a}{\sqrt{1+a^2}}. \tag{19}$$

For the angle of incidence defined by $\sin\alpha = U/c$ and Eq. (19), the charged particle has a drift velocity in (L'). When $\phi' = 0$, we can have $\hat{P}'_x = -a$, $\hat{P}'_y = 0$, and $\hat{P}'_z = 0$. If a is small enough, we have $\langle \hat{v}'_z \rangle \approx 3a'^2/4$. As a consequence, in the laboratory frame, the charged particle has the following average component along the z axis $\langle \hat{v}_z \rangle \approx 3a^2/4$.

B. Electromagnetic wave in a cold electron plasma

We now show that the propagation of a strong linearly polarized electromagnetic wave in a cold electron plasma generates a constant current along the propagation direction of the wave when the phase velocity of the wave is very close to the speed of light. It will be the case when the plasma density is much lower than the nonrelativistic critical density or when the wave is relativistically strong.

To describe the propagation of a relativistically strong wave in a cold electron plasma, we start from the Maxwell and Lorentz equations. All the variables entering into these equations are assumed not to be functions of space and time separately, but only of the combination $\mathbf{i}\cdot\mathbf{r} - Vt$, where $\mathbf{i}$ is a constant unit vector and V a constant. It means that we look for plane wave solutions traveling in the direction $\mathbf{i}$ with speed V. If we denote derivatives with respect to this quantity by a prime, the Maxwell equations can be written in the form

$$\mathbf{i} \times \mathbf{E}' = V\mathbf{B}', \tag{20a}$$

$$\mathbf{i} \times \mathbf{B}' = -\frac{V}{c^2}\mathbf{E}' - \mu_0 env, \tag{20b}$$

$$\mathbf{i} \cdot \mathbf{B}' = 0, \tag{20c}$$

$$\mathbf{i} \cdot \mathbf{E}' = -\frac{e}{\varepsilon_0}(n - N_0), \tag{20d}$$

$$(\mathbf{i}\cdot\mathbf{v}-V)p' = - = e\mathbf{E} - ev\times\mathbf{B}, \tag{20e}$$

where $\mathbf{v}$ and $\mathbf{p}$ are, respectively, the velocity and mechanical momentum of the electrons, n is their density, and N_0 is the one of ions.

By integrating Eq. (20a), one obtains

$$\mathbf{B} = V^{-1}(i\times\mathbf{E}). \tag{21}$$

Thus $\mathbf{E}$ and $\mathbf{B}$ are perpendicular.

Equations (20b) and (20d) lead to the following expression for the electron density:

$$n = \frac{N_0 V}{V - \mathbf{i}\cdot\mathbf{v}}. \tag{22}$$

Multiplying vectorially Eq. (20e) on the left by $\mathbf{i}$ and taking into account expression (21) give

$$\mathbf{B} = \frac{1}{e}(i\times\mathbf{p}'). \tag{23}$$

Multiplying Eq. (20b) vectorially on the left by $\mathbf{i}$ leads to

$$\mathbf{B}' = -\mu_0 en(\beta^2-1)^{-1}(\mathbf{i}\times\mathbf{v}), \tag{24}$$

where $\beta = V/c$.

Equations (23) and (24) allow us to derive a propagation equation for the electron momentum. Introducing the following variables

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{mc}, \quad \hat{v} = \frac{v}{c}, \quad \tau = t - \frac{\mathbf{i}\cdot\mathbf{r}}{V}, \quad \omega_p^2 = \frac{e^2 N_0}{\varepsilon_0 m}. \tag{25}$$

After some algebra the following equations for the electron motion are found [18]:

$$\frac{d^2\hat{p}_x}{d\tau^2} + \omega_p^2 \frac{\beta^2}{\beta^2-1}\,\frac{\beta\hat{p}_x}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z} = 0, \tag{26a}$$

$$\frac{d^2\hat{p}_y}{d\tau^2} + \omega_p^2 \frac{\beta^2}{\beta^2-1}\,\frac{\beta\hat{p}_y}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z} = 0, \tag{26b}$$

$$\frac{d^2}{d\tau^2}(\beta\hat{p}_z - \sqrt{1+\hat{p}^2}) + \omega_p^2 \frac{\beta^2\hat{p}_z}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z} = 0. \tag{26c}$$

Letting $\theta = \omega_p(\beta^2-1)^{-1/2}\tau$, these equations become [12]

$$\frac{d^2\hat{p}_x}{d\theta^2} + \frac{\beta^3\hat{p}_x}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z} = 0, \tag{27a}$$

$$\frac{d^2\hat{p}_y}{d\theta^2} + \frac{\beta^3\hat{p}_y}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z} = 0, \tag{27b}$$

$$\frac{d^2}{d\theta^2}(\beta\hat{p}_z - \sqrt{1+\hat{p}^2}) + \frac{\beta^2(\beta^2-1)\hat{p}_z}{\beta\sqrt{1+\hat{p}^2}-\hat{p}_z} = 0. \tag{27c}$$

Assuming that the phase velocity is close to the speed of light, i.e., $\beta\approx 1$, Eq. (27c) leads to the fact that the following quantity is a constant:

$$C = \alpha^2 = \sqrt{1+\hat{p}^2} - \hat{p}_z. \tag{28}$$

This invariant is the same as one of those found in the case of one particle in a linearly polarized wave.

As the wave is supposed to be linearly polarized in the *x-y* plane, it is considered that $\hat{p}_y = 0$. Thus, Eqs. (27a) and (27b) lead to

$$\hat{p}_x = \hat{p}_{0x}\cos(\theta/\alpha),$$

$$\hat{p}_y = 0, \tag{29}$$

where $\hat{p}_{0x}$ is a constant. Assuming that $\hat{p}_z = 0$ when $\theta = 0$, we have

$$\hat{C} = \alpha^2 = \sqrt{1+\hat{p}_{0x}^2}. \tag{30}$$

Equations (29) and (30) allow us to find the following expression for the frequency:

$$\omega_0 = \omega_p(\beta^2-1)^{-1/2}(1+\hat{p}_{0x}^2)^{-1/4}. \tag{31}$$

$\hat{p}_z$ can be calculated through Eq. (30)

$$\hat{p}_z = -\frac{\hat{p}_{0x}^2\sin^2(\theta/\alpha)}{2\sqrt{1+\hat{p}_{0x}^2}}. \tag{32}$$

Equation (23) gives

$$B_x = 0,$$

$$B_y = \frac{p_x'}{e} = \frac{m\omega_0}{\beta e}\hat{p}_{0x}\sin(\theta/\alpha),$$

$$B_z = 0. \tag{33}$$

From Eq. (21) the electric field is calculated

$$E_x = \frac{mc\omega_0}{e}\hat{p}_{0x}\sin(\theta/\alpha),$$

$$E_y = 0,$$

$$E_z = 0. \tag{34}$$

Thus, it is consistent to take $\hat{p}_{0x} = a$. The following expressions for the velocity $\hat{\mathbf{v}} = \hat{\mathbf{p}}/\gamma$ and $\gamma = 1/\sqrt{1-\hat{\mathbf{v}}^2}$ allow $\hat{\mathbf{v}} = \hat{\mathbf{p}}/\sqrt{1+\hat{\mathbf{p}}^2}$. Thus, Eq. (32) yields

$$\hat{v}_z = -\frac{a^2\sin^2[\theta/(1+a^2)^{1/4}]}{2+a^2\{1+\cos^2[\theta/(1+a^2)^{1/4}]\}}. \tag{35}$$

The average velocity obtained is the same as in the single particle case when $\hat{P}_x = 0$, $\hat{P}_y = 0$, and $\hat{P}_z = 0$ when $\phi = 0$. The values of $\langle\hat{v}_z\rangle$ calculated with Eq. (35) and the numerical solution of Eqs. (27) were compared for different values of β. For a given upper value of θ, the agreement improves when β goes to unity.

III. MOTION OF A CHARGED PARTICLE IN A LINEARLY POLARIZED ELECTROMAGNETIC TRAVELING WAVE PROPAGATING ALONG A CONSTANT HOMOGENEOUS MAGNETIC FIELD

The constant magnetic field $\mathbf{B}_0$ is supposed to be along the z axis. The traveling wave is assumed to be linearly polarized and to propagate through a medium with an index of refraction n. It has a propagation vector $\mathbf{k}_0$ parallel to $\mathbf{B}_0$. The fields are given by

$$E_x = E_0 \sin(\omega_0 t - k_0 z), \quad E_y = 0, \quad E_z = 0,$$
$$B_x = 0, \quad B_y = \frac{k_0 E_0}{\omega_0} \sin(\omega_0 t - k_0 z), \quad B_z = B_0. \tag{36}$$

The following vector potential is chosen for the electromagnetic field

$$\mathbf{A} = \left[\frac{E_0}{\omega_0} \cos(\omega_0 t - k_0 z) \right] \hat{e}_x + (B_0 x) \hat{e}_y. \tag{37}$$

The relativistic Hamiltonian for the motion is

$$H = \left[\left(P_x + \frac{eE_0}{\omega_0} \cos(\omega_0 t - k_0 z) \right)^2 c^2 + (P_y + eB_0 x)^2 c^2 + P_z^2 c^2 + m^2 c^4 \right]^{1/2}. \tag{38}$$

This is a time-dependent system with three degrees of freedom. P_y is a constant of motion as the Hamiltonian does not depend on y explicitly. It can be easily checked that $C = H - (\omega_0/k_0) P_z$ is still a constant of motion for this system. Moreover, combining the equations of Hamilton, it can be readily found that $I = P_x + eB_0 y$ is also a constant of motion. The following Poisson brackets: $[P_y, C] = 0$, $[C, I] = 0$, and $[P_y, I] = -eB_0$ show that these three constants are not in involution and one cannot conclude that the problem is integrable.

New dimensionless variables and a new dimensionless parameter are now introduced:

$$\hat{x} = k_0 x, \quad \hat{y} = k_0 y, \quad \Omega_0 = \frac{eB_0}{m\omega_0}, \quad \hat{H} = n\gamma = \frac{nH}{mc^2}. \tag{39}$$

As it is assumed that the electromagnetic wave propagates through a medium with an index of refraction $n(n = k_0 c/\omega_0)$, the normalized Hamiltonian expressed in terms of the dimensionless variables and parameters is

$$\hat{H} = n \left[[\hat{P}_x + a \cos(\hat{t} - \hat{z})]^2 + \left(\hat{P}_y + \frac{\Omega_0}{n} \hat{x} \right)^2 + \hat{P}_z^2 + 1 \right]^{1/2}. \tag{40}$$

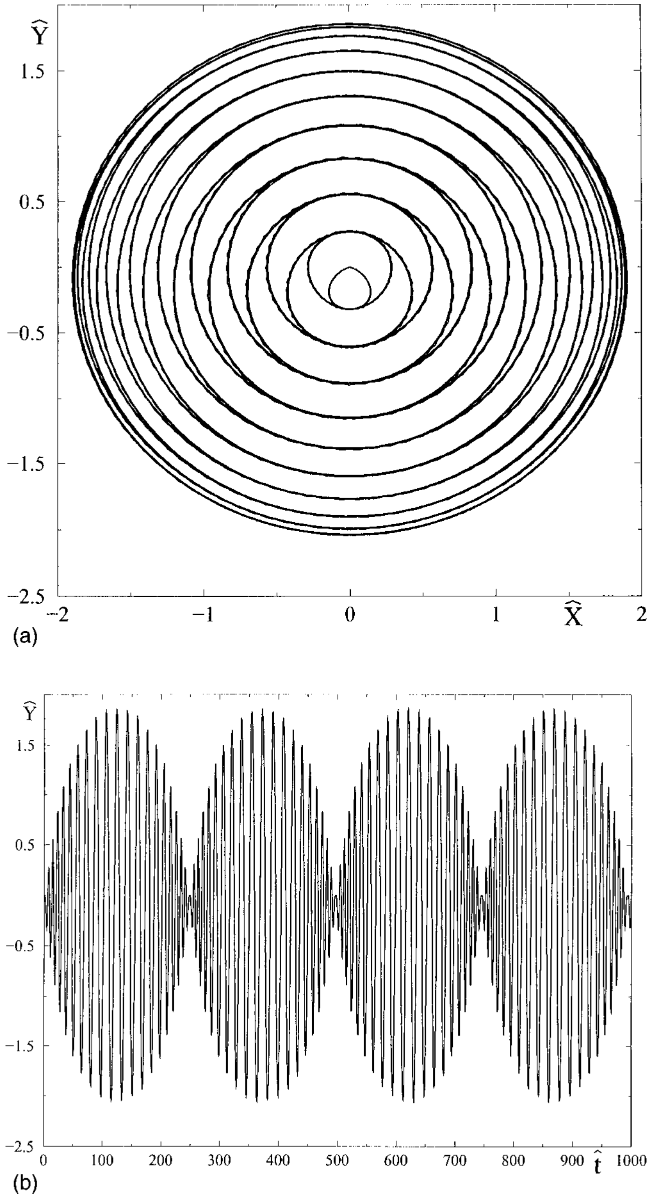

FIG. 1. (a) Trajectory of a charged particle initially nonresonant and at rest $\gamma_0 = 1$, $\Omega_0 = 1.05$ at $\hat{x}_0 = \hat{y}_0 = 0$ in the $\hat{x}$-$\hat{y}$ plane. $a = 10^{-1}$, $n = 1$. (b) $\hat{y}$ component of the charged particle's position in the same conditions as those of (a).

In the normalized variables, the constants of motion are $\hat{P}_y$, $\hat{C} = \hat{H} - \hat{P}_z$, and $\hat{I} = \hat{P}_x + \Omega_0 / n\hat{y}$. The canonical equations are solved numerically using a fourth order Runge-Kutta method. When the electromagnetic wave propagates in vacuum $(n = 1)$, two different types of trajectories are obtained. When the particle is initially at rest and resonant, it spirals outward in the plane perpendicular to the z axis. When it is initially nonresonant and at rest, it spirals outward and inward (Fig. 1). When the wave propagates in a medium with an index of refraction inferior to unity $(n < 1)$, the trajectories spiral outward and inward just like in the previous nonresonant case. This situation corresponds, for instance, to the case when a low intensity wave with a high frequency propagates in a plasma.

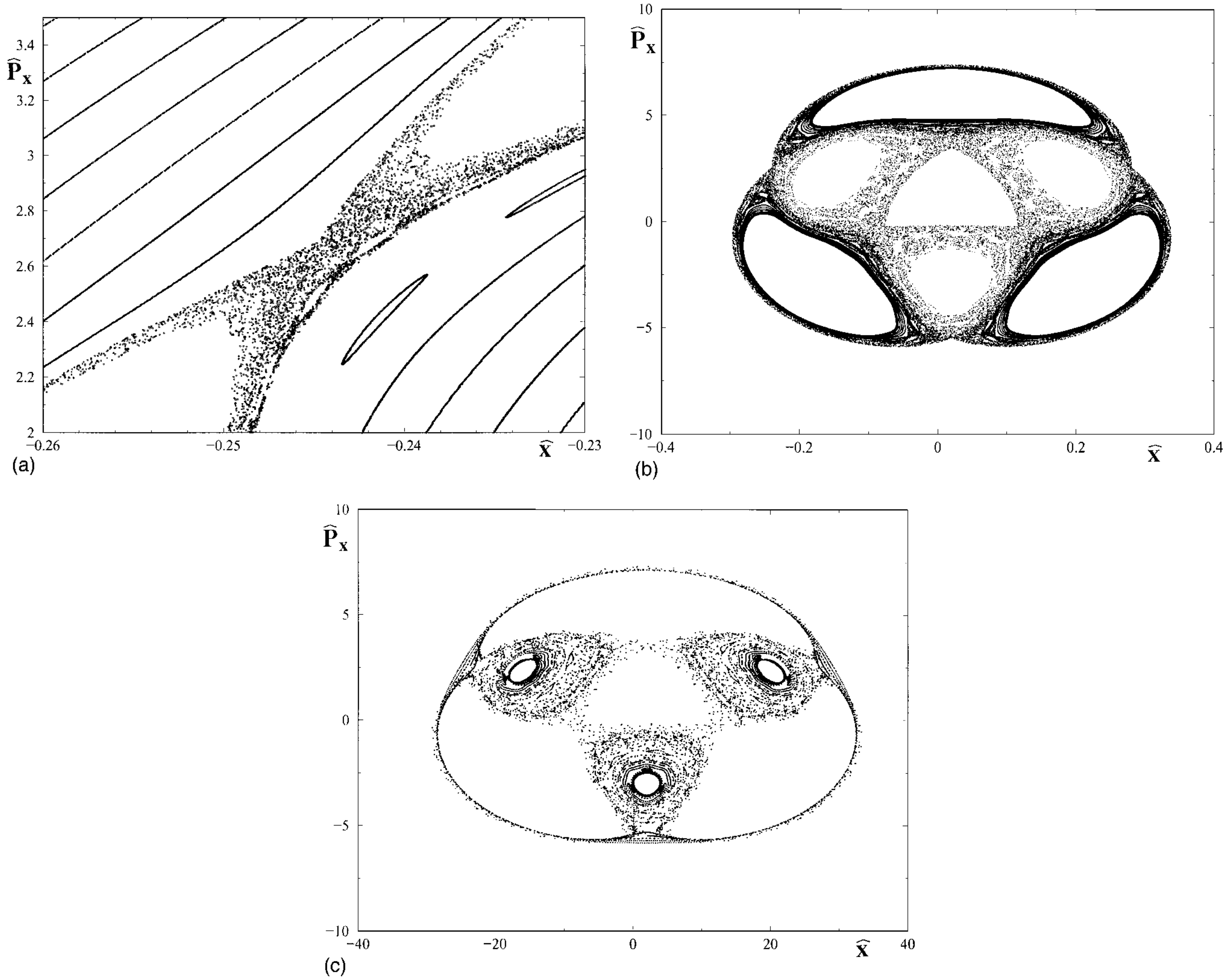

FIG. 2. (a) Surface of section plots for some trajectories. $a=3$, $\Omega_0=2.001$, and $n=0.1$. (b) Surface of section plots for some trajectories. $a=4.03$, $\Omega_0=2$, and $n=0.1$. (c) Surface of section plots for some trajectories. $a=4.03$, $\Omega_0=0.1997$, and $n=0.995$.

Then, the canonical transformation defined by Eq. (8) is performed. The Hamiltonian becomes

$$\overline{H}=n\left[(\hat{P}_x+a\cos\phi)^2+\left(\hat{P}_y+\frac{\Omega_0}{n}\hat{x}\right)^2+\hat{P}_z^2+1\right]^{1/2}-\hat{P}_z. \tag{41}$$

Note that this expression of the Hamiltonian is the constant $\hat{C}$ expressed in terms of the new variables. The two other constants are $\hat{P}_y$ and $\hat{I}=\hat{P}_x+(\Omega_0/n)\hat{y}$. The equations of Hamilton are

$$\dot{\hat{P}}_x=-\frac{\Omega_0}{\gamma}\left(\hat{P}_y+\frac{\Omega_0}{n}\hat{x}\right),$$

$$\dot{\hat{x}}=\frac{n}{\gamma}(\hat{P}_x+a\cos\phi),$$

$$\dot{\hat{P}}_y=0,$$

$$\dot{\hat{y}}=\frac{n}{\gamma}\left(\hat{P}_y+\frac{\Omega_0}{n}\hat{x}\right),$$

$$\dot{\hat{P}}_z=\frac{an}{\gamma}\sin\phi(\hat{P}_x+a\cos\phi), \tag{42}$$

$$\dot{\phi}=\frac{n\hat{P}_z}{\gamma}-1.$$

These equations are also solved numerically. Chaos is evidenced by performing Poincaré maps. The plane $\hat{P}_x-\hat{x}$ with $\phi=0$ (mod 2π) is chosen to be the Poincaré surface of section. Figures 2(a)–2(c) show Poincaré maps for several trajectories. As a consequence the system is not integrable.

Introducing the variables

$$\overline{X}=\hat{P}_x+a\cos\phi, \tag{43}$$

$$\overline{P_X}=\hat{P}_y+\frac{\Omega_0}{n}\hat{x},$$

and the complex quantity $Z=\overline{P_X}+i\overline{X}$, the two first equations of Hamilton [Eqs. (42)] are equivalent to the following equation:

$$\dot{Z}=-\frac{i\Omega_0}{\gamma}Z-ia\dot{\phi}\sin\phi, \tag{44}$$

which is the equation of a nonlinear oscillator under the action of an external force. Formally, the solution of this equation can be written

$$Z=A_0\exp-i[\sigma(\hat{t})+\delta]-\frac{a}{2}\int_0^{\hat{t}}\dot{\phi}(\tau)[\exp(-i\vartheta)-\exp(-i\chi)]d\tau, \tag{45}$$

where A_0 and δ are real constants, $\vartheta=\sigma(\hat{t})-\sigma(\tau)-\phi(\tau)$, $\chi=\sigma(\hat{t})-\sigma(\tau)+\phi(\tau)$, and

$$\sigma(\hat{t})=\Omega_0\int_0^{\hat{t}}d\tau\gamma^{-1}(\tau). \tag{46}$$

Then

$$\hat{x}=\frac{A_0 n}{\Omega_0}\cos[\sigma(\hat{t})+\delta]-\frac{an}{2\Omega_0}\int_0^{\hat{t}}\dot{\phi}(\tau)[\cos\vartheta-\cos\chi]d\tau-\frac{n}{\Omega_0}\hat{P}_y, \tag{47a}$$

$$\hat{P}_x=-A_0\sin[\sigma(\hat{t})+\delta]-a\cos\phi+\frac{a}{2}\int_0^{\hat{t}}\dot{\phi}(\tau)\times[\sin\vartheta-\sin\chi]d\tau. \tag{47b}$$

The quantities A_0 and δ are determined, so that, at $\hat{t}=0$, $A_0^2=\gamma_0^2-\hat{p}_{z0}^2-1=\hat{p}_{x0}^2+\hat{p}_{y0}^2$ and $\tan\delta=-\hat{p}_{x0}/\hat{p}_{y0}$ ($\hat{p}=p/mc$, p is the mechanical momentum of the particle). The subscript 0 appended to variables γ and p refers to their initial values. This formal solution shows that when the particle has an initial normalized energy γ_0 large compared to a ($A_0\gg a$), the trajectory of the particle is an ellipse in the $\hat{x}-\hat{P}_x$ plane. Equation (47b) is now substituted in the equation for $\dot{P}_z$ [Eqs. (42)] to obtain the following nonlinear integrodifferential equation:

$$\dot{\hat{P}}_z=-\frac{an}{\gamma}\sin\phi\left[A_0\sin(\sigma(\hat{t})+\delta)-\frac{a}{2}\int_0^{\hat{t}}\dot{\phi}(\tau)\times(\sin\vartheta-\sin\chi)d\tau\right]. \tag{48}$$

Let us now consider that the index of the medium is unity and that the particle is resonant. In terms of the original coordinates, the condition for resonance is

$$\omega_0-k_0\dot{z}-\frac{eB_0}{m\gamma}=0. \tag{49}$$

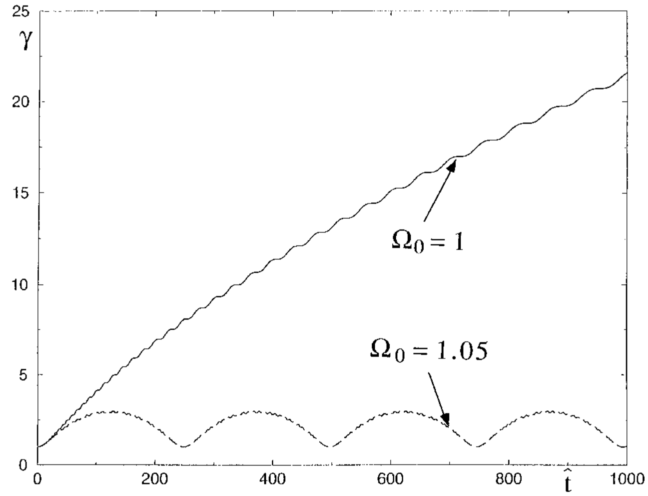

FIG. 3. γ vs time for two values of Ω_0. $\gamma_0=1$. $a=0.1$ and $n=1$.

This condition implies that $\hat{C}$ is such as $\hat{C}=eB_0/m\omega_0$. This means that if the particle is initially resonant, it remains resonant all the time [15]. It has been checked numerically in some cases that the synchronous solution is still possible in a linearly polarized wave (Figs. 3 and 4). In Fig. 3, the comparison of the energy of the particle versus time between the resonant case and a nonresonant one is shown. In Fig. 4, the values of γ obtained for one value of parameter a when the wave is linearly polarized are compared to those obtained when it is circularly polarized. In terms of the new coordinates, the resonance condition becomes

$$\dot{\phi}+\frac{\Omega_0}{\gamma}=0. \tag{50}$$

Integrating this expression from 0 to $\hat{t}$ gives

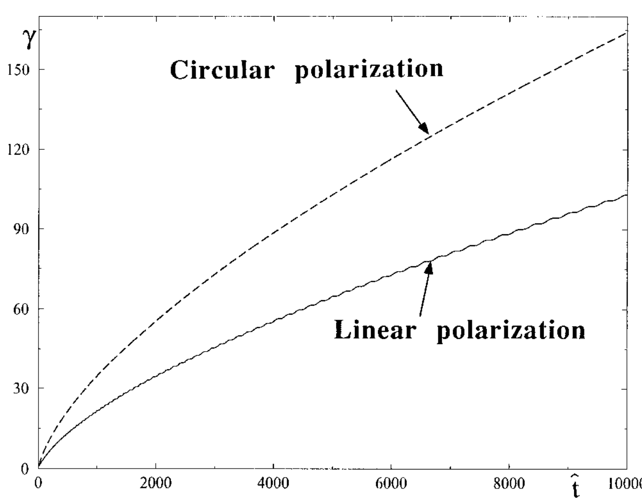

FIG. 4. Comparison of the evolution of γ between the case when the wave is circularly polarized and the case when it is linearly polarized for the same value of parameter a. $\gamma_0=\Omega_0=1$, $a=0.1$, and $n=1$.

$$\phi(\hat{t})+\sigma(\hat{t})-\phi_0=0, \tag{51}$$

where ϕ_0 is the value of ϕ at $\hat{t}=0$. At resonance, Eq. (48) becomes

$$\dot{\hat{P}}_z=-\frac{a}{\gamma}\sin\phi(\hat{t})\left[A_0\sin[\theta_0-\phi(\hat{t})]+\frac{a}{2}\int_0^{\hat{t}}\dot{\phi}(\tau)\{\sin\phi(\hat{t}) + \sin[2\phi(\tau)-\phi(\hat{t})]\}d\tau\right], \tag{52}$$

with $\theta_0=\phi_0+\delta$.
Assuming the charged particle is initially at rest and that $\phi_0=0$ at $\hat{t}=0$, taking into account that $\hat{C}$ is a constant, Eq. (52) yields

$$\dot{\gamma}=\frac{a^2}{2}\dot{\phi}\phi\sin^2\phi. \tag{53}$$

This equation is integrated between 0 and $\hat{t}$ to give

$$\gamma=1+\frac{a^2}{8}[\phi^2-\tfrac{1}{2}(\cos 2\phi+2\phi\sin 2\phi-1)], \tag{54}$$

while ϕ has to satisfy

$$\dot{\phi}=-\frac{1}{1+a^2/8[\phi^2-1/2(\cos 2\phi+2\phi\sin 2\phi-1)]}. \tag{55}$$

Integrating this equation we obtain

$$\frac{a^2}{24}\phi^3+\phi\left(1+\frac{a^2}{16}+\frac{a^2}{16}\cos 2\phi\right)-\frac{a^2}{16}\sin 2\phi+\hat{t}=0. \tag{56}$$

This equation can be solved numerically, and thus a function $\phi(\hat{t})$ can be built up. As a consequence, γ and all the variables of the system can be expressed in terms of quadratures. Considering that ϕ is large enough ($\phi \gg 1$) and a small compared to unity, the phase ϕ can be shown to scale as $\phi \approx -2\times 3^{1/3}a^{-2/3}\hat{t}^{1/3}$. By substituting this expression of ϕ into Eq. (55), the following scaling law for the Lorentz factor is obtained

$$\gamma\approx\frac{3^{2/3}}{2}a^{2/3}\hat{t}^{2/3}. \tag{57}$$

The good agreement between the evolution of γ versus time obtained with this expression and the one derived numerically through the exact equations of motion was verified numerically.

IV. CONCLUSIONS

Using the Hamiltonian formalism, we showed that, in a strong linearly polarized traveling wave, a charged particle has an average velocity along its propagation direction. In a cold electron plasma, the wave equations derived by Akhiezer and Polovin permit us to show that all the electrons can have this same constant velocity. Thus, there can be a constant current along the wave propagation direction when the phase velocity of the wave is very close to the speed of light, e.g., when the wave is relativistically strong and/or when the plasma density is much lower than the nonrelativistic critical density.

Again with the help of the Hamiltonian formalism, the problem of relativistic motion of a charged particle in a transverse linearly polarized traveling wave and a constant homogeneous magnetic field was studied. Only two constants of motion, independent and in involution were found. When the wave propagates in a vacuum, one of these first integrals appears in the resonance condition. It was verified numerically that the synchronous solution is still possible in a linearly polarized wave, as such a wave can be considered as the sum of two circularly polarized components. The nonintegrability was proven by performing Poincaré maps. We then transformed the system into an autonomous one by a canonical transformation and a formal solution was given for all the variables in terms of the energy of the particle and of the phase of the wave. Finally, assuming that the wave propagates in a vacuum, it was shown that when the particle is initially resonant and at rest, the system can be simply expressed in terms of quadratures allowing, in the case of low intensities, a scaling law for the charged particle energy, to be derived.

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